The Geometry of the Frame Bundle over Spacetime

Fredrik Ståhl*

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Abstract

One of the known mathematical descriptions of singularities in General Relativity is the b-boundary, which is a way of attaching endpoints to inextendible endless curves in a spacetime. The b-boundary of a manifold M with connection Γ is constructed by forming the Cauchy completion of the frame bundle LM equipped with a certain Riemannian metric, the b-metric G. We study the geometry of (LM, G) as a Riemannian manifold in the case when Γ is the Levi-Cività connection of a Lorentzian metric g on M. In particular, we give expressions for the curvature and discuss the isometries and the geodesics of (LM, G) in relation to the geometry of (M, g).

1 Introduction

In general relativity, the concept of singularities is unavoidable. Known solutions to Einstein's field equations display a variety of non-trivial singularities, giving rise to such diverse phenomena as black holes, the big bang and topological anomalies. The situation is very different from most other field theories, where singular solutions may be explained as artefacts of idealised modelling (like point charges) or differentiability restrictions (like shock waves), for example. In order to study singularities, it is important to have a mathematical machinery that allows us to treat questions like convergence or divergence of physical quantities when approaching the singularity. In other words, one would like to incorporate the singularities together with the regular spacetime points in some abstract set, equipped with a suitable topology that allows one to define statements such as 'close to the singularity' in a mathematically precise sense.

One definition of this kind is the b-boundary [16]. Given a manifold M with connection Γ, the b-boundary of M is formed by constructing a suitable metric, the b-metric G, on the bundle of frames LM. The pair (LM, G) can then be viewed as a Riemannian manifold, and in particular, as a topological metric

*Department of Mathematics, University of Umeå, S-901 87 Umeå, Sweden.
E-mail address: Fredrik.Stahl@math.umu.se

space. The b-boundary is formed by taking the Cauchy completion of (LM, G), and a projection then gives an extension $\overline{M}$ of the original manifold M. We leave the details to the next section.

The b-boundary construction has several drawbacks however, the most important being that the topology on the extended set $\overline{M}$ is non-Hausdorff in general (see, e.g., [4, 6, 18]). There has been some attempts to remedy the situation (see [7, 8]), although they have not been entirely successful.

In order to obtain a more complete understanding of what goes wrong with the b-boundary, we need to understand the geometry of (LM, G) better. This is the subject at hand. Since the object of interest is spacetime, we restrict attention to the case when Γ is the Levi-Cività connection of a Lorentzian metric g on M.

The outline of the paper is as follows: in §2, we go through the essential steps of the b-boundary definition. In §3 and §4, we calculate the connection and curvature of (LM, G) and discuss some of the implications. Finally, §5 and §6 are devoted to a discussion of the isometries and the geodesics of (LM, G) in relation to the corresponding structures on (M, g).

2 The b-boundary

We will attach an abstract boundary set to a Lorentzian manifold (M, g), where M is a smooth n-dimensional connected orientable Hausdorff manifold with a smooth metric g of signature $n-2$. The case of interest in relativity theory is of course when $n = 4$.

The construction of the b-boundary may be carried out in different bundles over M (see [16], [6] or [12] for some background). It is often convenient to work with the bundle of pseudo-orthonormal frames OM, consisting of all pseudo-orthonormal frames at all points of M. In our case this leads to complications because of the amount of algebra involved, so we choose instead to construct the b-boundary via the bundle of general linear frames LM.

LM is a principal fibre bundle with the general linear group $GL(n)$ on $\mathbb{R}^n$ as its structure group. We write the right action of an element $A \in GL(n)$ as $R_A : E \mapsto EA$ for $E \in LM$. If M is orientable, the frame bundle LM has two connected components which we denote by L^+M and L^-M, corresponding to frames with positive and negative orientation, respectively. We will use the notation $L'M$ for any one of these two components. Clearly, any $A \in GL(n)$ which changes the orientation sets up a 1–1 correspondence between L^+M and L^-M, and we may regard the component of the identity $GL'(n) \subset GL(n)$ as the structure group of $L'M$.

The fibre bundle structure of LM gives a canonical 1-form $\boldsymbol{\theta} : T(LM) \to \mathbb{R}^n$, and the connection corresponds to a connection form $\boldsymbol{\omega} : T(LM) \to \mathfrak{gl}(n)$, where $\mathfrak{gl}(n)$ is the Lie algebra of $GL(n)$ [14]. Let $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{gl}(n)}$ be Euclidian inner products with respect to fixed bases in $\mathbb{R}^n$ and $\mathfrak{gl}(n)$, respectively. We

define a Riemannian metric G on LM, the *b-metric* or *Schmidt metric*, by

$$G(X,Y) := \langle \boldsymbol{\theta}(X), \boldsymbol{\theta}(Y)\rangle_{\mathbb{R}^n} + \langle \boldsymbol{\omega}(X), \boldsymbol{\omega}(Y)\rangle_{\mathfrak{gl}(n)}. \tag{1}$$

G can be shown to be uniformly equivalent under a change of bases in $\mathbb{R}^n$ and $\mathfrak{gl}(n)$ [16, 6].

If γ is a curve in LM, the *b-length* of γ is the length of γ with respect to the b-metric G, and is denoted by $l(\gamma)$. Thus

$$l(\gamma) := \int \Big(|\boldsymbol{\theta}(\dot\gamma)|^2 + \|\boldsymbol{\omega}(\dot\gamma)\|^2 \Big)^{1/2} \mathrm{d}t, \tag{2}$$

where $|\cdot|$ and $\|\cdot\|$ are the fixed Euclidian norms in $\mathbb{R}^n$ and $\mathfrak{gl}(n)$, respectively, and $\dot\gamma$ is the tangent of γ.

If γ is horizontal, $\boldsymbol{\omega}(\dot\gamma) = 0$ and γ may be written as a pair (λ, E) where $\lambda = \pi \circ \gamma$ is a curve in M and E is the parallel frame along λ given by γ. By definition, $\boldsymbol{\theta}(\dot\gamma)$ is the vector of the components of the tangent $\dot\lambda$ in the frame E. So the b-length of a horizontal curve γ is equivalent to the length measured in a parallel frame along $\pi \circ \gamma$ (which is called 'generalised affine parameter length' in [12]). This motivates further study of the b-metric, since the presence of an endless curve with finite generalised affine parameter length is often taken as the criterion for a spacetime to be singular.

Following Schmidt [16], we now use the b-metric G to construct a topological boundary of the base manifold M, providing endpoints for all endless curves with finite b-length. Since $(L'M, G)$ is a Riemannian manifold, it is a metric space with topological metric d (the b-metric distance function), and so the Cauchy completion $\overline{L'M}$ of $L'M$ is well defined. $\overline{L'M}$ is a complete metric space, and we define the boundary of $L'M$ as $\partial L'M := \overline{L'M} \setminus L'M$.

The topological metric d then has a unique extension $\bar d$ to $\overline{L'M}$. It can be shown that the action of $GL'(n)$ is uniformly continuous on $(L'M, G)$, viewed as a metric space [16, 6]. It follows that there is a unique uniformly continuous extension of the right action of $GL'(n)$ to $(\overline{L'M}, G)$.

Justified by the above, we may now construct the topological space

$$\overline{M} := \overline{L'M}/GL'(n), \tag{3}$$

the set of orbits of $GL'(n)$ in $\overline{L'M}$. We can also define a continuous projection $\bar\pi : \overline{L'M} \to \overline{M}$ taking a point in $\overline{L'M}$ to the corresponding orbit of $GL'(n)$. By definition, $\bar\pi$ coincides with π on $L'M$. Thus $\bar\pi(L'M) = M$ may be regarded as a subset of $\overline{M}$, and we define the *b-boundary* of M as $\partial M := \overline{M} \setminus M$.

It is important to emphasise that the topological structure of $\overline{L'M}$ may be quite complicated. In particular, in many relevant cases $\overline{L'M}$ is non-Hausdorff [1, 13, 4, 18]. This is related to the possibility of 'fibre degeneracy', as the fibre bundle structure usually cannot be extended to $\overline{L'M}$. A boundary orbit may even be a single point [3, 18].

For the rest of this paper we will, somewhat sloppily, write LM instead of $L'M$ and $GL(n)$ instead of $GL'(n)$.

3 Cartan's equations and the connection

It is clear from the definition (1) of G that

$$E^i := \boldsymbol{\theta}^i \qquad \text{together with} \qquad E^{\{i,i'\}} := \boldsymbol{\omega}^i_{i'} \tag{4}$$

is a global orthonormal coframe field on (LM, G). If we let uppercase latin indices take values in the extended index set $\{i = 1 \dots n\} \cup \{\{i, i'\};\ i, i' = 1 \dots n\}$, then E^I is a basis for the cotangent space $T^*(LM)$. Note that the brackets denote ordered pairs and not unordered sets. If we denote the connection form of (LM, G) by $\tilde{\boldsymbol{\omega}}$, applying Cartan's first equation with vanishing torsion gives

$$\mathrm{d}E^I + \tilde{\boldsymbol{\omega}}^I{}_J \wedge E^J = 0. \tag{5}$$

We can also apply Cartan's equations to $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$ on M, which gives

$$\mathrm{d}\boldsymbol{\theta}^i + \boldsymbol{\omega}^i_j \wedge \boldsymbol{\theta}^j = 0 \tag{6}$$

and

$$\boldsymbol{\Omega}^i_j = \mathrm{d}\boldsymbol{\omega}^i_j + \boldsymbol{\omega}^i_k \wedge \boldsymbol{\omega}^k_j, \tag{7}$$

where $\boldsymbol{\Omega}^i_j = \frac{1}{2} R^i_{jkl}\, \boldsymbol{\theta}^k \boldsymbol{\theta}^l$ is the curvature 2-form and R^i_{jkl} are the frame components of the Riemann tensor of (M, g). Combining (5) with (6) and (7) we get the system

$$\begin{aligned} \tilde{\boldsymbol{\omega}}^i_J \wedge E^J &= \boldsymbol{\omega}^i_j \wedge \boldsymbol{\theta}^j, \\ \tilde{\boldsymbol{\omega}}^{\{i,i'\}}_J \wedge E^J &= \boldsymbol{\omega}^i_j \wedge \boldsymbol{\omega}^j_{i'} - \tfrac{1}{2} R^i_{i'jk}\, \boldsymbol{\theta}^j \boldsymbol{\theta}^k. \end{aligned} \tag{8}$$

Since G is a Riemannian metric, the metric condition is

$$\tilde{\boldsymbol{\omega}}^I{}_J = -\tilde{\boldsymbol{\omega}}^J{}_I. \tag{9}$$

Solving (8–9) for $\tilde{\boldsymbol{\omega}}$ we get

$$\begin{aligned} \tilde{\boldsymbol{\omega}}^i_j &= \tfrac{1}{2}(\boldsymbol{\omega}^i_j - \boldsymbol{\omega}^j_i) - \tfrac{1}{2} \sum_{k,l} R^k{}_{lij} \boldsymbol{\omega}^k{}_l, \\ \tilde{\boldsymbol{\omega}}^i{}_{\{j,j'\}} &= -\tilde{\boldsymbol{\omega}}^{\{j,j'\}}{}_i = -\tfrac{1}{2}(\delta^i_j \boldsymbol{\theta}^{j'} + \delta^i_{j'} \boldsymbol{\theta}^j) - \tfrac{1}{2} \sum_k R^j_{j'ik} \boldsymbol{\theta}^k, \\ \tilde{\boldsymbol{\omega}}^{\{i,i'\}}{}_{\{j,j'\}} &= \tfrac{1}{2}\Big[\delta^i_j(\boldsymbol{\omega}^{i'}{}_{j'} - \boldsymbol{\omega}^{j'}{}_{i'}) + \delta^{i'}{}_{j'}(\boldsymbol{\omega}^i_j - \boldsymbol{\omega}^j_i) + \delta^{i'}{}_j \boldsymbol{\omega}^i_{j'} - \delta^i_{j'} \boldsymbol{\omega}^j_{i'}\Big]. \end{aligned} \tag{10}$$

We have left the index positions fixed and written out the summations explicitly to avoid confusing the two metrics involved.

4 The curvature

Our objective is now to compute the curvature of (LM, G) expressed in the basis E^I. If we denote the curvature form on LM by $\tilde{\boldsymbol{\Omega}}$, Cartan's second equation on LM gives

$$\tilde{\boldsymbol{\Omega}}^I{}_J = \mathrm{d}\tilde{\boldsymbol{\omega}}^I{}_J + \tilde{\boldsymbol{\omega}}^I{}_K \wedge \tilde{\boldsymbol{\omega}}^K{}_J. \tag{11}$$

In order to calculate $\mathrm{d}\tilde{\boldsymbol{\omega}}$ we first need an expression for $\mathrm{d}R^i_{jkl}$, which is given by the following lemma. The result is not new of course (see, e.g., [2]), but we still provide a proof for completeness.

Lemma 1. *The exterior derivatives of the frame components of the Riemann tensor, viewed as functions on LM, are*

$$\mathrm{d}R^i_{jkl} = -R^m{}_{jkl}\boldsymbol{\omega}^i{}_m + R^i_{mkl}\boldsymbol{\omega}^m{}_j + R^i_{jml}\boldsymbol{\omega}^m{}_k + R^i_{jkm}\boldsymbol{\omega}^m{}_l + R^i_{jkl;m}\boldsymbol{\theta}^m. \tag{12}$$

Here $R^i_{jkl;m}$ denote the frame components of the covariant derivative of the Riemann tensor of (M, g).

Proof. Let E_i be the standard horizontal vector fields dual to $\boldsymbol{\theta}^i$, fixed by the choice of basis for $\mathbb{R}^n$. Similarly, let $E_{\{i,i'\}}$ be the fundamental vertical vector fields dual to $\boldsymbol{\omega}^i_{i'}$, corresponding to the choice of basis for $\mathfrak{gl}(n)$. Clearly $E_I = \{E_i, E_{\{i,i'\}}\}$ is a basis for $T(LM)$. Moreover, it is the unique dual of E^I. From the definition of the exterior derivative,

$$\mathrm{d}R^i_{jkl} = \sum_m E_m(R^i_{jkl})\,\boldsymbol{\theta}^m + \sum_{m,m'} E_{\{m,m'\}}(R^i_{jkl})\,\boldsymbol{\omega}^m{}_{m'}. \tag{13}$$

To proceed further we introduce coordinates on LM as follows. Let $p \in M$ and let x^i be coordinates for M on a neighbourhood $\mathcal{U}$ of p. Given a frame $F = (F_i)$ at a point $q \in \mathcal{U}$, we may express each F_i as

$$F_i = \Big(\frac{\partial}{\partial x^a}\Big) X^a_i. \tag{14}$$

The determinant of the matrix X is nonzero, so we may use (x^a, X^b_i) as coordinates for LM on $\pi^{-1}(\mathcal{U})$.

The coordinate expressions for the horizontal vector fields E_i and the vertical vector fields $E_{\{i,i'\}}$ are then

$$E_i = \Big(\frac{\partial}{\partial x^a} - \Gamma^b_{ac} X^c_j \frac{\partial}{\partial X^b_j}\Big) X^a_i, \tag{15}$$

$$E_{\{i,i'\}} = \Big(\frac{\partial}{\partial X^a_{i'}}\Big) X^a_i, \tag{16}$$

where Γ^b_{ac} are the Christoffel symbols of (M, g) in the coordinates x^a [14]. Now the Riemann tensor frame components R^i_{jkl} are related to the coordinate components R^a_{bcd} by

$$R^i_{jkl} = (X^{-1})^i_a R^a_{bcd} X^b_j X^c_k X^d_l, \tag{17}$$

where X^{-1} is the inverse of X. Applying (15–16) to (17) and inserting into (13) then gives the desired result. □

With the help of (10), (6–7) and Lemma 1, it is possible to solve (11) for the curvature form $\tilde{\mathbf{\Omega}}$, and then calculate the Riemann tensor, the Ricci tensor and the curvature scalar. It is a trivial but long and tedious exercise, so we will not describe it here. Some of the results are given in Appendix A, here we just give the expression for the curvature scalar $\tilde{R}$:

$$\tilde{R} = -\tfrac{1}{2}n^2(n+3) - \tfrac{1}{4}\sum_{i,j,k,l}(R^i_{jkl})^2 + \sum_i R_{ii}, \tag{18}$$

where R^i_{jkl} and R_{ij} are the frame components of the Riemann and Ricci tensor of (M, g), respectively.

What is the relevance of (18) in relation to singularities? A b-incomplete endless curve λ in M has an endpoint p on the b-boundary ∂M. The horizontal lift γ of λ has finite b-length and ends at ∂LM. In many cases, some frame component of the Riemann tensor will diverge along γ (c.f. [12, 4, 5, 9]). Using the terminology of [9], we say that p is a *curvature singularity.* We treat the case when the divergence is unbounded.

Proposition 2. *Let γ be a horizontal curve ending at ∂LM, and suppose that some frame component of the Riemann tensor of (M, g) tends to $\pm\infty$ along γ. Then the curvature scalar $\tilde{R}$ of (LM, G) tends to $-\infty$ along γ.*

Proof. The last term in (18) may be written as

$$\sum_{i,j} R^i_{jij}, \tag{19}$$

which is clearly dominated by the second term if some $R^i_{jkl} \to \pm\infty$. □

We now turn to the case when the frame components of the Riemann tensor are bounded on a curve ending at the boundary point. In this case the obstruction to extending M as a spacetime could be either some oscillatory divergence of the curvature, or purely topological. One might ask if it is possible that (LM, G) can be extended as a Riemannian manifold even if (M, g) is inextendible. We answer this question in the case when the boundary fibre is totally degenerate, in a sense which we now specify.

The b-boundary construction outlined in §2 can also be carried out using the bundle of pseudo-orthonormal frames OM. OM is a fibre bundle with the Lorentz group $\mathcal{L}$ as its structure group, and there is a natural inclusion $OM \subset LM$ such that OM is a reduced subbundle of LM [14]. The b-metric is defined on OM by (1), i.e., exactly in the same way as for LM. Since the connection in OM is simply the reduction of the connection in LM, (OM, G) is a Riemannian submanifold of (LM, G). We write $\overline{OM}$ and ∂OM for the Cauchy

completion and boundary of OM, respectively. Then an alternative definition (see [6, 10]) of the b-boundary is $\partial M := \overline{M} \setminus M$ with

$$\overline{M} := \overline{OM}/\mathcal{L}. \tag{20}$$

In [18], it was shown that for many exact solutions in general relativity, the 'fibre' (or, more correctly, the orbit of the extended group action) over a point $p \in \partial M$ in OM is totally degenerate, i.e., a single point. Since $\partial OM \subset \partial LM$, the fibre in LM is degenerate as well, though possibly not completely.

Proposition 3. *Suppose that (LM, G) is (locally) extendible through $\bar{\pi}^{-1}(p)$, where $p \in \partial M$, and that the corresponding boundary fibre in $\overline{OM}$ is completely degenerate. Then (M, g) is asymptotically a conformally flat Einstein space, i.e. the curvature R of (M, g) is given by $R^i_{jkl} = \frac{1}{6} R g^i_{[k} g_{l]\,j}$ in the limit at p.*

Proof. Let $\lambda \colon [0,1] \to \overline{OM} \subset \overline{LM}$ be a horizontal curve ending at $\bar{\pi}^{-1}(p)$ such that the restriction to $[0, 1)$ is contained in OM. Since (LM, G) is extendible through $\bar{\pi}^{-1}(p)$, the curvature scalar $\tilde{R}$ must have a well defined limit along $\lambda(t)$ as $t \to 1$. The restriction of $\bar{\pi}^{-1}(p)$ to $\overline{OM}$ is a single point, so the limit of $\tilde{R}$ must be invariant under the action of any Lorentz transformation $L \in \mathcal{L}$, changing the curve according to $\lambda \mapsto \lambda L$.

Now the curvature scalar $\tilde{R}$ is given by (18), which may be rewritten as

$$\tilde{R} = -\tfrac{1}{2} n^2 (n+3) - \tfrac{1}{4} I + R - 2 \sum_{\alpha,\beta,\gamma} (R^1{}_{\alpha\beta\gamma})^2 + 2R_{11}, \tag{21}$$

where 1 is the timelike index, greek indices go from 2 to n, R is the curvature scalar and I is the scalar invariant $R^{ijkl} R_{ijkl}$ of (M, g).

Since I and R are scalar invariants, we only need to consider the last two terms in (21). Given the frame E at $\lambda(t)$, we apply Lorentz transformations in the $n-1$ timelike planes spanned by E_1 and E_α, $\alpha = 2, 3, \ldots, n$. After some algebra we find that (21) is invariant if and only if

$$R^1{}_{\alpha\beta\gamma} = R^\beta{}_{\alpha 1 \gamma}, \tag{22}$$

$$R^1{}_{\alpha 1 \beta} = R^\gamma{}_{\alpha\gamma\beta}, \tag{23}$$

$$R^1{}_{\alpha 1 \beta} = 0 \qquad \text{if } \alpha \neq \beta, \tag{24}$$

$$R^\alpha{}_{\beta\gamma\epsilon} = 0 \qquad \text{if } \alpha \notin \{\beta, \gamma, \epsilon\}. \tag{25}$$

From (23) and (24), the Ricci tensor is given by

$$R_{ij} = \tfrac{1}{4} R g_{ij}, \tag{26}$$

which is the condition for an Einstein space [17]. Applying the first Bianchi identity to (22) and permuting the indices shows that $R^1{}_{\alpha\beta\gamma} = 0$. Thus (22–25) implies

$$R^i_{jkl} = \tfrac{1}{6} R g^i_{[k} g_{l]\,j}, \tag{27}$$

which means that the Weyl tensor vanishes. □

5 Isometries

5.1 Horizontal isometries

We seek isometries of (LM, G) which have horizontal orbits. Any transformation φ of M induces an automorphism $\tilde{\varphi}$ of LM taking a frame $E = (E_1, E_2, \ldots, E_n)$ at $p \in M$ to the frame $\tilde{\varphi} = (\varphi_* E_1, \varphi_* E_2, \ldots, \varphi_* E_n)$ at $\varphi(p) \in M$. A transformation of (M, g) is said to be *affine* if it preserves the connection. Obviously, the group of affine transformations includes the isometry group. From [14] we have the following.

Proposition 4.
(1) *For any φ, $\tilde{\varphi}$ leave the canonical 1-form $\boldsymbol{\theta}$ invariant. Moreover, any fibre-preserving automorphism of LM that leaves $\boldsymbol{\theta}$ invariant is induced by a transformation of M.*
(2) *The fibre-preserving automorphisms of LM that leaves both the canonical 1-form $\boldsymbol{\theta}$ and the connection form $\boldsymbol{\omega}$ invariant are exactly those that are induced by the affine transformations of (M, g).*

It is worth noting that $\tilde{\varphi}$ maps OM into itself if and only if φ is an isometry of (M, g) [14]. From Proposition 4 we draw the following conclusion which is apparent from the definition (1) of the b-metric.

Corollary 5.
Any affine transformation φ of (M, g) induces an isometry $\tilde{\varphi}$ of (LM, G).

There might of course be other isometries of (LM, G) induced by non-affine transformations of (M, g). They are characterised by the property that they preserve the inner product $\langle \boldsymbol{\omega}, \boldsymbol{\omega} \rangle_{\mathfrak{gl}(n)}$. This includes transformations that change the connection form according to $\boldsymbol{\omega} \mapsto A\,\boldsymbol{\omega} B$ where A and B belong to the rotation group $O(n)$.

5.2 Vertical isometries

Our objective is now to find all vertical isometries of (LM, G), i.e., the isometries where each orbit is contained in a single fibre of LM. Assume that $V = V^I E_I$ is a vertical Killing vector field. Then the corresponding covector V_I is given by

$$V_i = 0 \qquad \text{and} \qquad V_{\{i,i'\}} = a_{ii'}, \tag{28}$$

for some function a from LM to the Lie algebra $\mathfrak{gl}(n)$, and satisfies the Killing equation

$$V_{(I;J)} = 0. \tag{29}$$

Here the semicolon denotes a covariant derivative with respect to G and the parentheses denote symmetrisation.

From the expressions (10) for the connection form $\tilde{\boldsymbol{\omega}}$ we may identify the connection coefficients in the frame E on LM. Then (29) becomes

$$a_{(ij)} = 0, \tag{30}$$

$$E_i(a_{jk}) = 0, \tag{31}$$

$$E_{\{i,j\}}(a_{kl}) + E_{\{k,l\}}(a_{ij}) - a_{(ik)}\delta_{jl} + a_{(jl)}\delta_{ik} = 0. \tag{32}$$

From (30), a is skew, and from (31), a is a function of the fibre coordinates only. Loosely speaking, a is the same on all fibres (locally). Thus (32) gives

$$E_{\{i,j\}}(a_{kl}) + E_{\{k,l\}}(a_{ij}) = 0. \tag{33}$$

This equation is obviously fulfilled if a is a constant element in $\mathfrak{o}(n)$, the Lie algebra of the rotation group $O(n)$. Then V is the Killing vector field of a global isometry generated by the right action R_A of $A = \exp a \in O(n)$. (This can of course be seen directly from the definition (1) of the b-metric, using the transformation properties of $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$ under the action R_A.)

Instead of searching for non-constant solutions to (33), we reformulate the problem as finding local isometries of a single fibre $\mathcal{F}$, which is warranted by (31). Since $\mathcal{F}$ is isomorphic to $GL(n)$, we may view a_{ij} as the Killing vector field of a local isometry of $GL(n)$ with the metric $G_{\mathcal{F}}$ induced by G. By the definition of the connection form $\boldsymbol{\omega}$, $G_{\mathcal{F}}$ is invariant under the *left* action of $GL(n)$, but not necessarily under the right action. To study this in more detail, we introduce coordinates on $\mathcal{F}$ as in §4. Let x^i be coordinates in a neighbourhood of the base point $\pi(\mathcal{F}) \in M$. Then any frame $F = (F_i) \in \mathcal{F}$ can be expressed by $F_i = (\frac{\partial}{\partial x^a})X^a{}_i$ for some $X \in GL(n)$, and the map $\varphi\colon \mathcal{F} \to GL(n)$ given by $F \mapsto X$ is an isomorphism. The restriction of the connection form $\boldsymbol{\omega}$ to the fibre is

$$\boldsymbol{\omega}^i{}_j = (X^{-1})^i{}_k \, \mathrm{d}X^k{}_j, \tag{34}$$

and $G_{\mathcal{F}}$ is given by

$$G_{\mathcal{F}}(X, Y) = \langle \boldsymbol{\omega}(X), \boldsymbol{\omega}(Y) \rangle_{\mathfrak{gl}(n)}. \tag{35}$$

Apparently, we may view $G_{\mathcal{F}}$ as a metric on $GL(n)$. The invariance of $G_{\mathcal{F}}$ under left translations $X \to AX$ is apparent from (34). But we also see that $G_{\mathcal{F}}$ is not invariant under the right action of $GL(n)$.

Under φ, the orthonormal basis $E_{\{i,j\}}$ of $T(\mathcal{F})$ is mapped to the orthonormal basis e_{ij} of $\mathfrak{gl}(n)$ used when defining the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{gl}(n)}$. From the fact that a is skew and (33) we get

$$(e_{ij} + e_{ji})(a_{kl}) = 0, \tag{36}$$

which means that a is completely specified by its value on $O(n) \subset GL(n)$. Also, from (34) and (35), the restriction of $G_{\mathcal{F}}$ to $O(n)$ is invariant under both left and right translations of $O(n)$. Since $O(n)$ is a compact Lie group, there is essentially only one choice of a bi-invariant metric on $O(n)$ [11]. We have thus arrived at a complete characterisation of the vertical isometries of (LM, G).

Proposition 6. *The group of (local) vertical isometries of (LM, G) is isomorphic to the group of (local) isometries of $O(n)$ equipped with the 'canonical' bi-invariant metric.*

6 Geodesics

Suppose that γ is a geodesic of (LM, G), with tangent vector $v^i E_i + V^{ij} E_{\{i,j\}}$ and affine parameter t. Reading off the connection coefficients from (10) and inserting into the geodesic equation gives

$$\dot{v}^i = \sum_j V^{ji} v^j + \sum_{j,k,l} R^k_{\ lij} V^{kl} v^j, \tag{37}$$

$$\dot{V}^{ii'} = -v^i v^{i'} + \sum_k \big(V^{ki} V^{ki'} - V^{ik} V^{i'k} \big), \tag{38}$$

where the dot denotes an ordinary derivative with respect to t. Note that the right hand side of (38) is symmetric, which means that the skew part of $V^{ii'}$ must be constant.

6.1 Vertical geodesics

As we saw in §5.2, any fibre $\mathcal{F}$ with the metric $G_{\mathcal{F}}$ induced by G is isometric to $GL(n)$ with the metric generated by the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{gl}(n)}$. Thus the vertical geodesics are simply the geodesics of $GL(n)$ with respect to this metric, so the vertical geodesics are not coupled to the geometry of (M, g) at all. Still, we clarify a few points.

If $v^i \equiv 0$, (37) is automatically satisfied and (38) becomes

$$\dot{V}^{ii'} = \sum_k \big(V^{ki} V^{ki'} - V^{ik} V^{i'k} \big). \tag{39}$$

We see that if V is constant and skew, it satisfies (39). In other words, orbits of R_A are geodesics if and only if $A \in O(n)$. This is not surprising since $O(n)$ is a compact Lie group, and $G_{\mathcal{F}}$ corresponds to the canonical bi-invariant metric on $O(n)$. Thus $(O(n), G_{\mathcal{F}})$ is a normal homogeneous space, and it is a property of such spaces that the geodesics are given by orbits of right actions [11]. Note that this means that $(O(n), G_{\mathcal{F}})$ is a totally geodesic submanifold of $(GL(n), G_{\mathcal{F}})$.

Constant symmetric V are also solutions of (39). Although it looks non-linear, (39) actually has a simple structure. Let W and C be the symmetric and skew part of V, respectively. As noted above, C must be constant, so (39) becomes

$$\dot{W}^{ii'} = -2 \sum_k \big(C^{ik} W^{ki'} + C^{i'k} W^{ki} \big), \tag{40}$$

i.e., a system of linear ordinary differential equations in W.

6.2 Horizontal geodesics

If γ is horizontal, $V^{ii'} \equiv 0$ along γ. It follows from (38) that $v^i \equiv 0$, so there are no horizontal geodesics of (LM, G). In particular, no geodesic of (LM, G) is a horizontal lift of a geodesic of (M, g).

In [19], properties of curves γ in M with extremal length was studied, with the length measured in a parallel pseudo-orthonormal frame along γ. It was found that for an extremal γ to be a geodesic of (M, g), it is necessary that

$$\delta_{ij} W^i R^j{}_{klm} W^k W^l = 0, \tag{41}$$

where W^i is the tangent vector of γ and $R^j{}_{klm}$ are the components of the Riemann tensor, both expressed in the parallel frame. This is a severe restriction on (M, g).

Locally, geodesics may be considered as extremal curves of the length functional. We can reformulate the result in [19] as follows: even if we restrict attention to horizontal curves in OM, an extremal of the b-length functional cannot be expected to be a lift of a geodesic in (M, g).

7 Discussion

There are many open questions as to the structure of (LM, G) for physically interesting spacetimes (M, g). In fact, even for the 2-dimensional Schwarzschild spacetime, it is not known if the b-boundary is of dimension 0 or 1. Hopefully, geometric methods in (LM, G) may help to shed some light on the situation. But it should be remembered that actual calculations with the b-boundary is difficult even in the two-dimensional case.

Proposition 2 also raises some questions. Loosely speaking, that $\tilde{R} \to -\infty$ may be interpreted as that the geometry becomes 'infinitely hyperbolic' at the boundary point. However, not all sectional curvatures have to diverge, which means that the 'circumference' of the boundary point may be finite in some planes while it is infinite in other planes.

Also, the geodesics of (LM, G) may be very complicated in general, with no obvious connection to the geodesics of (M, g). The geodesic equations (37–38) suggest the possibility of a kind of oscillatory behaviour of the horizontal components v^i of the tangent vector.

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The calculations of the connection and curvature forms were verified with a modified version of the *Ricci* package [15] for *Mathematica* (Wolfram Research, Inc).

A Expressions for the curvature

As described in §3, the curvature form $\tilde{\mathbf{\Omega}}$ of (LM, G) can be found by applying (6–7) and Lemma 1 to (10). Because of the different metrics involved, some care is needed to keep track of which metric is used for contractions and raising and lowering of indices. Note that covariant and contravariant components in the frame $E_I = (\boldsymbol{\theta}^i, \boldsymbol{\omega}^i_{i'})$ may be identified, since the frame components of G is given by

$$G_{ij} = \delta_{ij}, \quad G_{i\{j,j'\}} = G_{\{j,j'\}i} = 0 \quad \text{and} \quad G_{\{i,i'\}\{j,j'\}} = \delta_{ij}\delta_{i'j'}. \tag{42}$$

So we may use δ, the Kronecker delta, for index operations. We also use the notation $\boldsymbol{\theta}^i$, $\boldsymbol{\omega}^i_j$ and R^i_{jkl} for the components of the canonical form $\boldsymbol{\theta}$, the connection form $\boldsymbol{\omega}$ and the curvature tensor R of (M, g) in the frame on (M, g) specified by the location in LM. If we raise and lower indices with δ, an ambiguity arises when applying symmetries and contractions to the Riemann tensor. For example, R_{ijkl} will *not* be equal to R_{jikl} since the first index is lowered with δ instead of g. Therefore we keep the index positions fixed and adopt the convention that all repeated indices represent contractions, regardless of their variance. We content ourselves with showing the results, since the calculations involve a substantial amount of algebra.

The components of the curvature form $\tilde{\mathbf{\Omega}}$ in the frame $E_I = (\boldsymbol{\theta}^i, \boldsymbol{\omega}^i_{i'})$ are

$$\begin{aligned}
\tilde{\mathbf{\Omega}}^i_{\ j} = {} & \tfrac{1}{2}\,\boldsymbol{\theta}^i \wedge \boldsymbol{\theta}^j \\
& + \tfrac{1}{4}\left(R^i_{lkj} - R^j_{lki} + R^k_{lij} - R^{mm'}{}_{ik}R_{mm'jl} - R^{mm'}{}_{ij}R_{mm'kl}\right)\boldsymbol{\theta}^k \wedge \boldsymbol{\theta}^l \\
& - \tfrac{1}{2}\,R^l_{l'ij;k}\,\boldsymbol{\theta}^k \wedge \boldsymbol{\omega}^l_{l'} - \tfrac{1}{4}\left(\boldsymbol{\omega}^i_{\ k} + \boldsymbol{\omega}^k_{\ i}\right) \wedge \left(\boldsymbol{\omega}^j_{\ k} + \boldsymbol{\omega}^k_{\ j}\right) \\
& + \tfrac{1}{2}\,R^k_{lij}\left(\boldsymbol{\omega}^m_{\ k} \wedge \boldsymbol{\omega}^m_{\ l} + \boldsymbol{\omega}^k_{\ m} \wedge \boldsymbol{\omega}^m_{\ l} + \boldsymbol{\omega}^k_{\ m} \wedge \boldsymbol{\omega}^l_{\ m}\right) \\
& + \tfrac{1}{4}\,R^k_{lim}\boldsymbol{\omega}^k_{\ l} \wedge \left(\boldsymbol{\omega}^m_{\ j} + \boldsymbol{\omega}^j_{\ m}\right) + \tfrac{1}{4}\,R^k_{ljm}\boldsymbol{\omega}^k_{\ l} \wedge \left(\boldsymbol{\omega}^m_{\ i} + \boldsymbol{\omega}^i_{\ m}\right) \\
& - \tfrac{1}{4}\,R^k_{k'im}R^l_{l'jm}\,\boldsymbol{\omega}^k_{k'} \wedge \boldsymbol{\omega}^l_{l'},
\end{aligned} \tag{43}$$

$$\begin{aligned}
\tilde{\mathbf{\Omega}}^i_{\ \{j,j'\}} = -\tilde{\mathbf{\Omega}}^{\{j,j'\}}{}_i = {} & \tfrac{1}{2}\,R^j_{j'ik;l}\,\boldsymbol{\theta}^k \wedge \boldsymbol{\theta}^l \\
& + \tfrac{1}{4}\left(\boldsymbol{\theta}^{j'} \wedge \boldsymbol{\omega}^j_{\ i} - \boldsymbol{\theta}^j \wedge \boldsymbol{\omega}^i_{\ j'} - \delta_{ij}\,\boldsymbol{\theta}^k \wedge (\boldsymbol{\omega}^{j'}{}_k + \boldsymbol{\omega}^k_{\ j'}) - \delta_{ij'}\,\boldsymbol{\theta}^k \wedge \boldsymbol{\omega}^k_{\ j}\right) \\
& - \tfrac{1}{4}\left(R^l_{l'ij'}\,\boldsymbol{\theta}^j + R^l_{l'ij}\,\boldsymbol{\theta}^{j'}\right) \wedge \boldsymbol{\omega}^l_{l'} - \tfrac{1}{4}\,R^j_{j'kl}\,\boldsymbol{\theta}^k \wedge \left(\boldsymbol{\omega}^i_{\ l} + \boldsymbol{\omega}^l_{\ i}\right) \\
& - \tfrac{1}{4}\,R^l_{j'ik}\boldsymbol{\theta}^k \wedge \left(\boldsymbol{\omega}^j_{\ l} + \boldsymbol{\omega}^l_{\ j}\right) + \tfrac{1}{4}\,R^j_{lik}\,\boldsymbol{\theta}^k \wedge \left(\boldsymbol{\omega}^{j'}{}_l + \boldsymbol{\omega}^l_{\ j'}\right) \\
& + \tfrac{1}{4}\,R^{j'}{}_{lik}\,\boldsymbol{\theta}^k \wedge \boldsymbol{\omega}^j_{\ l} - \tfrac{1}{4}\,R^l_{jik}\,\boldsymbol{\theta}^k \wedge \boldsymbol{\omega}^l_{\ j'} - \tfrac{1}{4}\,R^l_{l'im}R^j_{j'mk}\,\boldsymbol{\theta}^k \wedge \boldsymbol{\omega}^l_{l'},
\end{aligned} \tag{44}$$

$$
\begin{aligned}
\tilde{\boldsymbol{\Omega}}^{\{i,i'\}}{}_{\{j,j'\}} &= -\tfrac{1}{4}\big(\delta_{i'j'}\,\boldsymbol{\theta}^i\wedge\boldsymbol{\theta}^j + \delta_{i'j}\,\boldsymbol{\theta}^i\wedge\boldsymbol{\theta}^{j'} + \delta_{ij'}\,\boldsymbol{\theta}^{i'}\wedge\boldsymbol{\theta}^j + \delta_{ij}\,\boldsymbol{\theta}^{i'}\wedge\boldsymbol{\theta}^{j'}\big) \\
&\quad + \tfrac{1}{4}\left(R^i_{i'j'l}\,\boldsymbol{\theta}^j + R^i_{i'jl}\,\boldsymbol{\theta}^{j'} - R^j_{j'i'l}\,\boldsymbol{\theta}^i - R^j_{j'il}\,\boldsymbol{\theta}^{i'}\right)\wedge\boldsymbol{\theta}^l \\
&\quad + \tfrac{1}{4}\left(\delta_{i'j'}(R^i_{jkl} - R^j_{ikl}) + \delta_{ij}(R^{i'}{}_{j'kl} - R^{j'}{}_{i'kl}) + \delta_{i'j}R^i_{j'kl} - \delta_{j'i}R^j_{i'kl}\right)\boldsymbol{\theta}^k\wedge\boldsymbol{\theta}^l \\
&\quad - \tfrac{1}{4}\,R^i_{i'km}R^j_{j'lm}\,\boldsymbol{\theta}^k\wedge\boldsymbol{\theta}^l + \tfrac{1}{4}\left(\boldsymbol{\omega}^i_j\wedge\boldsymbol{\omega}^{i'}{}_{j'} - \boldsymbol{\omega}^j_i\wedge\boldsymbol{\omega}^{j'}{}_{i'}\right) \qquad (45) \\
&\quad - \tfrac{1}{4}\,\delta_{ij}\left((\boldsymbol{\omega}^{i'}{}_k + \boldsymbol{\omega}^k_{i'})\wedge(\boldsymbol{\omega}^k_{j'} + \boldsymbol{\omega}^{j'}{}_k) + \boldsymbol{\omega}^k_{i'}\wedge\boldsymbol{\omega}^k_{j'}\right) \\
&\quad - \tfrac{1}{4}\,\delta_{i'j'}\left((\boldsymbol{\omega}^i_k + \boldsymbol{\omega}^k_i)\wedge(\boldsymbol{\omega}^k_j + \boldsymbol{\omega}^j_k) + \boldsymbol{\omega}^i_k\wedge\boldsymbol{\omega}^j_k\right) \\
&\quad - \tfrac{1}{4}\,\delta_{ij'}\left(\boldsymbol{\omega}^{i'}{}_k\wedge\boldsymbol{\omega}^j_k + \boldsymbol{\omega}^k_{i'}\wedge\boldsymbol{\omega}^k_j\right) - \tfrac{1}{4}\,\delta_{i'j}\left(\boldsymbol{\omega}^i_k\wedge\boldsymbol{\omega}^{j'}{}_k + \boldsymbol{\omega}^k_i\wedge\boldsymbol{\omega}^k_{j'}\right).
\end{aligned}
$$

From (43–45) we can obtain the components of the Riemann tensor, and a contraction then gives the Ricci tensor:

$$\tilde{R}_{ij} = -\,\delta_{ij} - \tfrac{1}{2}\,R^k{}_{k'il}R^k{}_{k'jl} + R_{ij}, \tag{46}$$

$$\tilde{R}_{i\{j,j'\}} = \tilde{R}_{\{j,j'\}i} = \tfrac{1}{2}\,R^j_{j'ik;k}, \tag{47}$$

$$\tilde{R}_{\{i,i'\}\{j,j'\}} = -\,\tfrac{2n+1}{2}\,\delta_{ij'}\delta_{i'j} - \tfrac{n+1}{2}\,\delta_{ij}\delta_{i'j'} + \tfrac{3}{2}\,\delta_{ii'}\delta_{jj'} + \tfrac{1}{4}\,R^i_{i'kk'}R^j_{j'kk'}. \tag{48}$$

Contracting again gives the curvature scalar $\tilde{R}$ as given in §4, equation (18).

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